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Table of Laplace Transforms

Remember that we consider all functions (signals) as defined only on $t \geq 0$.

General

$f(t)$	$F(s) = \int_0^\infty f(t) e^{-st}\, dt$
$f + g$	$F + G$
$\alpha f \ (\alpha \in \mathbf{R})$	αF
$\frac{df}{dt}$	$sF(s) - f(0)$
$\frac{d^k f}{dt^k}$	$s^k F(s) - s^{k-1} f(0) - s^{k-2} \frac{df}{dt}(0) - \cdots - \frac{d^{k-1} f}{dt^{k-1}}(0)$
$g(t) = \int_0^t f(\tau)\, d\tau$	$G(s) = \frac{F(s)}{s}$
$f(\alpha t)$, $\alpha > 0$	$\frac{1}{\alpha} F(s/\alpha)$
$e^{at} f(t)$	$F(s - a)$
$t f(t)$	$-\frac{dF}{ds}$
$t^k f(t)$	$(-1)^k \frac{d^k F(s)}{ds^k}$
$\frac{f(t)}{t}$	$\int_s^\infty F(s)\, ds$
$g(t) = \begin{cases} 0 & 0 \leq t < T \\ f(t-T) & t \geq T \end{cases}$, $T \geq 0$	$G(s) = e^{-sT} F(s)$

Specific

1	$\frac{1}{s}$
δ	1
$\delta^{(k)}$	s^k
t	$\frac{1}{s^2}$
$\frac{t^k}{k!},\ k \geq 0$	$\frac{1}{s^{k+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2} = \frac{1/2}{s-j\omega} + \frac{1/2}{s+j\omega}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2} = \frac{1/2j}{s-j\omega} - \frac{1/2j}{s+j\omega}$
$\cos(\omega t + \phi)$	$\frac{s\cos\phi - \omega\sin\phi}{s^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$

Notes on the derivative formula at $t = 0$

The formula $\mathcal{L}(f') = sF(s) - f(0_-)$ must be interpreted very carefully when f has a discontinuity at $t = 0$. We'll give two examples of the correct interpretation.

First, suppose that f is the constant 1, and has no discontinuity at $t = 0$. In other words, f is the constant function with value 1. Then we have $f' = 0$, and $f(0_-) = 1$ (since there is no jump in f at $t = 0$). Now let's apply the derivative formula above. We have $F(s) = 1/s$, so the formula reads

$$\mathcal{L}(f') = 0 = sF(s) - 1$$

which is correct.

Now, let's suppose that g is a unit step function, *i.e.*, $g(t) = 1$ for $t > 0$, and $g(0) = 0$. In contrast to f above, g has a jump at $t = 0$. In this case, $g' = \delta$, and $g(0_-) = 0$. Now let's apply the derivative formula above. We have $G(s) = 1/s$ (exactly the same as F!), so the formula reads

$$\mathcal{L}(g') = 1 = sG(s) - 0$$

which again is correct.

In these two examples the functions f and g are the same except at $t = 0$, so they have the same Laplace transform. In the first case, f has no jump at $t = 0$, while in the second case g does. As a result, f' has no impulsive term at $t = 0$, whereas g does. As long as you keep track of whether your function has, or doesn't have, a jump at $t = 0$, and apply the formula consistently, everything will work out.