AN EXPRESSION OF CERTAIN KNOWN FUNCTIONS AS GENERALIZED HYPERGEOMETRIC FUNCTIONS.

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§ 1. *Introduction.*

It has long been known that many functions (*e. g.*, the logarithm and the Legendre functions) are particular cases of the hypergeometric function, and that other functions (*e. g.*, the exponential function and the Bessel functions) are limiting cases of the hypergeometric function. The object of the present paper is to show that the latter class of functions is much larger than has hitherto been supposed; that, in fact, it includes (in addition to those functions already known to belong to it) the following five types of functions, namely:

1. The functions which arise in harmonic analysis in connection with the parabolic cylinder.
2. The error function, which arises in connection with the theories of probability, errors of observations, refraction, and conduction of heat.
3. The incomplete gamma function, studied by Legendre, Hočevar, Schlömilch, Prym, and Vallier.
4. The logarithm integral, discussed by Euler, Soldner, Gauss, Bessel, Laguerre, and Stieltjes, which is applied in the expression of the number of primes less than a given number.
5. The cosine integral, studied by Schlömilch and Besso.

The first result obtained in the present paper is that *all the functions above named can be derived by specialization and transformation from a single new function.*

This function is denoted by $W_{k,m}(z)$, and the various functions above mentioned arise from it by taking special relations between the parameters k and m, and transforming the functions thus obtained: thus, the functions of the parabolic cylinders are derived by taking $m = \frac{1}{4}$, the incomplete gamma function is derived by taking $k + m = -\frac{1}{2}$, and so on. It is further shown that the Bessel functions are also cases of the same function $W_{k,m}(z)$.

It is evident, therefore, that if a general theory can be found for the function $W_{k,m}(z)$, then all the properties of the special functions in question can be deduced as mere corollaries. This is effected in § 3 of the present paper, where it is shown that *the function $W_{k,m}(z)$ is really a limiting case of the hypergeometric function* (some of the exponents of the hypergeometric function becoming infinite in a certain way), and that it possesses a theory which can be derived from that of the hypergeometric function. It appears, however, that the function $W_{k,m}(z)$ has, in the course of the limit process by which it is obtained, acquired certain new properties which are not possessed by the parent hypergeometric function; these properties are given.

The general family of functions $W_{k,m}(z)$ includes other classes of functions besides the five types of known functions already mentioned, and attention is drawn to some of these classes which have not hitherto been investigated.

§ 2. *Expression of the Parabolic Cylinder Functions, Error Function, Incomplete Gamma Functions, Logarithm Integral, and Cosine Integral by means of a Single Function.*

We now proceed to define a function in terms of which the various known functions already mentioned can be expressed. The function in question will be denoted by $W_{k,m}(z)$, and will be defined for all values of z (real or complex) by the equation

$$W_{k,m}(z) =$$

$$\frac{\Gamma(k+\frac{1}{2}-m)}{2\pi} e^{-\frac{1}{2}z+\frac{1}{2}i\pi} z^k \int (-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t}dt,$$

where the path of integration begins at $t=+\infty$, and after encircling the point $t=0$ in the counter-clockwise direction returns to $t=+\infty$ again. The t-plane is supposed to be dissected by a cut from $t=0$ to $t=+\infty$, and the expression $(-t)^{-k-\frac{1}{2}+m}$ is defined to mean

$$e^{(-k-\frac{1}{2}+m)\log(-t)},$$

where the real value of $\log(-t)$ is taken when t is real and negative.

When the real part of $-k-\frac{1}{2}+m$ is positive, the path of integration can be deformed so as to coincide with the real axis in the t-plane, and then (as is easily seen) the complex integral can be replaced by the real integral

$$W_{k,\,m}(z) =$$
$$\frac{1}{\Gamma(-k+\frac{1}{2}+m)} z^k e^{-\frac{1}{2}z} \int_0^\infty t^{-k-\frac{1}{2}+m} \left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t} dt.$$

We shall now show that the functions of the parabolic cylinder, error function, etc., can be expressed in terms of this function $W_{k,\,m}(z)$.

1. *The functions of the parabolic cylinder.*

The standard function associated with the parabolic cylinder in harmonic analysis is *

$$D_n(z) = \frac{2^{\frac{1}{2}n-2}\,\Gamma\left(\dfrac{n+1}{2}\right)}{\pi i^n} z \int e^{-\frac{1}{4}z^2 t} (t-1)^{-\frac{1}{2}n-\frac{1}{2}} (t+1)^{\frac{1}{2}n} dt,$$

where the integration is taken along a path which begins and ends at infinity in the direction which makes z^2t positive, and which encircles the point $t = 1$. This can be written

$$z^{\frac{1}{4}} D_{2k-\frac{1}{2}}(\sqrt{2z})$$
$$= \frac{2^{k-7/4}\Gamma(k+\frac{1}{4})}{\pi i^{2k-\frac{1}{2}}} z^{\frac{3}{4}} \int e^{-\frac{1}{2}zt} (t-1)^{-k-\frac{1}{4}} (t+1)^{k-\frac{1}{4}} dt,$$

and on writing $t = 1 + 2u/z$ in the integral and comparing with the definition of the function $W_{k,m}(z)$, we have

$$z^{\frac{1}{4}} D_{2k-\frac{1}{2}}(\sqrt{2z}) = 2^{k-\frac{1}{4}} W_{k,\frac{1}{4}}(z).$$

It follows that *the parabolic cylinder functions are expressed in terms of the function $W_{k,m}(z)$ by this equation,* and in fact correspond to the particular case $m = \frac{1}{4}$.

2. *The error function.*

The error function is defined by the integral

$$\int_z^\infty e^{-t^2} dt.$$

This function can be expressed in terms of the function $W_{k,m}(z)$. For from the definition of $W_{k,m}(z)$ we have

$$W_{-\frac{1}{4},\frac{1}{4}}(z^2) = z^{\frac{1}{2}} e^{-\frac{1}{2}z^2} \int_0^\infty \left(1+\frac{t}{z^2}\right)^{-\frac{1}{2}} e^{-t} dt$$
$$= 2z^{3/2} e^{-\frac{1}{2}z^2} \int_1^\infty e^{z^2(1-v^2)} dv, \qquad \text{where } v^2 = 1 + \frac{t}{z^2},$$

* Cf. a paper by the author in *Proc. Lond. Math. Soc.*, vol. 35, p. 417.

$$= 2z^{\frac{1}{2}}e^{\frac{1}{2}z^2}\int_z^\infty e^{-s^2}ds, \qquad \text{where } s = vz.$$

It follows that *the error function*

$$\int_z^\infty e^{-t^2}\,dt$$

can be expressed in terms of the function $W_{k,m}(z)$ *in the form*

$$\tfrac{1}{2}z^{-\frac{1}{2}}e^{-\frac{1}{2}z^2}W_{-\frac{1}{4},\frac{1}{4}}(z^2).$$

Certain other integrals discussed by Riemann in connection with the conduction of heat, *e. g.*, the integral

$$\int_a^b e^{-t^2-z^2/t^2}dt$$

can be evaluated in terms of the error function, and so in terms of the function $W_{k,m}(z)$.

3. *The incomplete gamma function.*

The incomplete gamma function is denoted by $\gamma(n, z)$, and is defined by the equation

$$\gamma(n, z) = \int_0^z t^{n-1}e^{-t}dt.$$

In order to express this in terms of the function $W_{k,m}(z)$, we observe that from the definition of the latter function we have

$$W_{\frac{1}{2}(n-1),\frac{1}{2}n}(z) = z^{\frac{1}{2}(n-1)}e^{-\frac{1}{2}z}\int_0^\infty\left(1+\frac{t}{z}\right)^{n-1}e^{-t}dt$$

$$= z^{\frac{1}{2}(1-n)}e^{\frac{1}{2}z}\int_z^\infty s^{n-1}e^{-s}ds, \qquad \text{where } s = t + z,$$

$$= z^{\frac{1}{2}(1-n)}e^{\frac{1}{2}z}\{\Gamma(n) - \gamma(n, z)\}.$$

It follows that *the incomplete gamma function can be expressed in terms of the function* $W_{k,m}(z)$ *by the relation*

$$\gamma(n, z) = \Gamma(n) - z^{\frac{1}{2}(n-1)}e^{-\frac{1}{2}z}W_{\frac{1}{2}(n-1),\frac{1}{2}n}(z).$$

4. *The logarithm integral.*

The logarithm integral $li(z)$ is defined by the equation

$$li(z) = \int_0^z \frac{dt}{\log t}.$$

The name was given by Soldner, and tables of the function were given by Bessel and Gauss.

Writing $z = e^{-\xi}$ and $t = e^{-u}$, we have

$$li(e^{-\xi}) = \int_{\infty}^{\xi} \frac{e^{-u}du}{u}$$

$$= -\frac{e^{-\xi}}{\xi}\int_0^{\infty}\left(1+\frac{s}{\xi}\right)^{-1} e^{-s}ds, \text{ where } s = u - \xi.$$

But

$$W_{-\frac{1}{2},0}(\xi) = \xi^{-\frac{1}{2}}e^{-\frac{1}{2}\xi}\int_0^{\infty}\left(1+\frac{t}{\xi}\right)^{-1} e^{-t}dt.$$

Therefore

$$li(e^{-\xi}) = -\xi^{-\frac{1}{2}}e^{-\frac{1}{2}\xi}W_{-\frac{1}{2},0}(\xi),$$

or *the logarithm integral is expressed in terms of the function* $W_{k,m}(z)$ *by the relation*

$$li(z) = -(-\log z)^{-\frac{1}{2}}z^{\frac{1}{2}}W_{-\frac{1}{2},0}(-\log z).$$

5. *The cosine integral.*

The cosine integral $Ci(z)$ is defined by the equation

$$Ci(z) = \int_{\infty}^{z}\frac{\cos t}{t}\,dt.$$

It can be expressed in terms of the function $W_{k,m}(z)$ in the following way. We have

$$Ci(z) = \tfrac{1}{2}\int_{\infty}^{z}\frac{e^{it}dt}{t} + \tfrac{1}{2}\int_{\infty}^{z}\frac{e^{-it}dt}{t}.$$

Putting $t = is$ in the first of these integrals, and $t = -is$ in the second, and remembering that

$$\int_{\pm i\infty}^{\infty}\frac{e^{-s}ds}{s}$$

is zero when the path of integration is wholly at infinity, we have

$$Ci(z) = \tfrac{1}{2}\int_{\infty}^{-iz}\frac{e^{-s}ds}{s} + \tfrac{1}{2}\int_{\infty}^{-iz}\frac{e^{-s}ds}{s}$$

$$= \tfrac{1}{2}li(e^{iz}) + \tfrac{1}{2}li(e^{-iz})$$

$$= \tfrac{1}{2}z^{-\frac{1}{2}}e^{\frac{1}{2}iz+i\pi/4}W_{-\frac{1}{2},0}(-iz) + \tfrac{1}{2}z^{-\frac{1}{2}}e^{-\frac{1}{2}iz-i\pi/4}W_{-\frac{1}{2},0}(iz),$$

which expresses the cosine integral in terms of the function $W_{k,m}(z)$.

6. It may be added that the Bessel functions can without much difficulty be shown to be likewise expressible in terms of the function $W_{k,m}(z)$. The actual relation will be found to be

$$J_m(z) = (2\pi z)^{-\frac{1}{2}}\{e^{(m+\frac{1}{2})i\pi/2}W_{0,m}(2iz) + e^{-(m+\frac{1}{2})i\pi/2}W_{0,m}(-2iz)\}.$$

§ 3. *Properties of the Function* $W_{k,m}(z)$.

We shall now establish the principal properties of the function $W_{k,m}(z)$.

1. *The differential equation satisfied by the function* $W_{k,m}(z)$. If we write

$$v = \int t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t}dt,$$

where the integration is taken along a path which begins at $t=\infty$, encircles the point $t=0$, and returns to $t=\infty$ again, we have

$$\begin{aligned}\frac{d^2v}{dz^2} &+ \left(\frac{2k}{z}-1\right)\frac{dv}{dz} + \frac{\frac{1}{4}-m^2+k(k-1)}{z^2}v \\ &= \frac{k-\frac{1}{2}+m}{z^2}\int t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-5/2+m} e^{-t}\left\{t\left(1+\frac{t}{z}\right)\right. \\ &\qquad\qquad \left.+ k-\tfrac{1}{2}-m+(1-2m)\frac{t}{z}\right\}dt \\ &= -\frac{k-\frac{1}{2}+m}{z^2}\int d\left\{t^{-k+\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{m-3/2+k} e^{-t}\right\};\end{aligned}$$

and this is zero, since the integrand is the perfect differential of a function which is zero at the terminal points of the path of integration. It follows that the function v satisfies the differential equation

$$\frac{d^2v}{dz^2} + \left(\frac{2k}{z}-1\right)\frac{dv}{dz} + \frac{\frac{1}{4}-m^2+k(k-1)}{z^2}v = 0.$$

Now from the definitions of $W_{k,m}(z)$ and of v, we have

$$W_{k,m}(z) = \text{constant} \times e^{-\frac{1}{2}z}z^k v,$$

and on substituting this value for v in the differential equation we see that $W_{k,m}(z)$ *satisfies the equation*

$$\frac{d^2W}{dz^2} + \left(-\tfrac{1}{4} + \frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2}\right) W = 0.$$

It may be noted that any differential equation of the form

$$\frac{d^2y}{dz^2} + \left(a + \frac{b}{z} + \frac{c}{z^2}\right) y = 0$$

can be brought to this form, and therefore solved in terms of the functions $W_{k,m}(z)$.

2. *Reduction of the function* $W_{k,m}(z)$ *to a limiting case of the hypergeometric function.*

The functions $W_{k,m}(z)$ have in general an irregular singularity (using the word "irregular" in the sense customary in the theory of linear differential equations) at $z = \infty$, and are therefore not members of the family of hypergeometric functions, which are characterized by the fact that their singularities are all regular. In spite of this, however, the functions $W_{k,m}(z)$ can be derived from the hypergeometric function by a limit process, namely, making the exponents at two of its singularities infinite, and making the two singularities in question to coalesce at infinity.

For the differential equation corresponding to the hypergeometric function

$$P\left\{\begin{matrix} 0 & \infty & c & \\ \tfrac{1}{2} + m & -c & c - k & z \\ \tfrac{1}{2} - m & 0 & k & \end{matrix}\right\}$$

is easily found to tend, for $c = \infty$, to the limiting form

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} + \left[\frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2}\right] y = 0,$$

and on putting $y = e^{-\frac{1}{2}z} v$ in this, we obtain

$$\frac{d^2v}{dz^2} + \left[-\tfrac{1}{4} + \frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2}\right] v = 0,$$

which is the differential equation satisfied by the function $W_{k,m}(z)$.

3. *Other solutions of the differential equation of the function* $W_{k,m}(z)$.

It is clear that the differential equation of the function $W_{k,m}(z)$ is unchanged if m be replaced by $-m$; it is also unchanged if k be replaced by $-k$, provided z be replaced by $-z$ at the same time. Hence *the four functions*

$$W_{k,m}(z), \quad W_{k,-m}(z), \quad W_{-k,m}(-z), \quad W_{-k,-m}(-z),$$

are solutions of the differential equation

$$\frac{d^2y}{dz^2} + \left(-\tfrac{1}{4} + \frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2}\right) y = 0.$$

Cases of this result are the well-known theorems that $J_n(z)$ and $J_{-n}(z)$ each satisfy Bessel's equation, and that $D_n(z)$ and $D_{-n-1}(iz)$ each satisfy the equation of the parabolic cylinder.

4. *The asymptotic expansion of* $W_{k,m}(z)$.

We have

$$W_{k,m}(z) =$$

$$\frac{\Gamma(k + \frac{1}{2} - m)}{2\pi} e^{-\frac{1}{2}z + \frac{1}{2}i\pi} z^k \int (-t)^{-k-\frac{1}{2}+m} \left(1 + \frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t} dt,$$

and it is easily seen that the divergent series which is obtained by expanding the quantity $(1 + t/z)^{k-\frac{1}{2}+m}$ by the binomial theorem, and then integrating term by term, is the asymptotic expansion of the integral for large positive values of z. This series is

$$e^{-\frac{1}{2}z} z^k \left\{1 + \frac{m^2 - (k - \frac{1}{2})^2}{1!\, z} \right.$$
$$\left. + \frac{\{m^2 - (k - \frac{1}{2})^2\}\{m^2 - (k - \frac{3}{2})^2\}}{2!\, z^2} + \cdots \right\}.$$

This is therefore the asymptotic expansion of the function $W_{k,m}(z)$.

The well-known asymptotic expansions of the logarithm integral, error function and Bessel functions, and the asymptotic expansion of the parabolic cylinder functions given by the author in *Proceedings of the London Mathematical Society*, volume 35, page 417, are cases of this expansion.

5. *Further properties of the functions* $W_{k,m}(z)$.

The remaining principal properties of functions of the type $W_{k,m}(z)$ will be stated without proof, as the reader will without difficulty be able to supply demonstrations.

(*a*) *A second definite integral expression for* $W_{k,m}(z)$ is

$$W_{k,m}(z) =$$

$$\tfrac{1}{2}\pi\Gamma(k+\tfrac{1}{4}+m)e^{-\frac{1}{2}z+i\pi/2}z^{k}\int(-t)^{-k-\frac{1}{2}-m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}-m}e^{-t}dt,$$

the integration being taken as before along a path which begins and ends at $t = +\infty$ and encircles $t = 0$.

(*b*) The function $W_{k,m}(z)$ *degenerates into an elementary function* when $k - \frac{1}{2} \pm m$ is a positive integer or zero; the theorem that the Bessel function degenerates into an elementary function when its order is half an odd integer is evidently a particular case of this result.

(*c*) The function $W_{k,m}(z)$ *degenerates into the indefinite integral of an elementary function* when $k - \frac{1}{2} \pm m$ is a negative integer; the incomplete gamma function, logarithm integral, error function, and cosine integral, are all instances of this theorem.

(*d*) Considering the case in which $W_{k,m}(z)$ degenerates into an elementary function, if $W_{k_1,m}(z)$ and $W_{k_2,m}(z)$ are two functions of this kind, with different parameters k but the same parameter m, *the integral property*

$$\int_0^\infty W_{k_1,m}(z)\,W_{k_2,m}(z)\frac{dz}{z} = 0$$

can be established.

(*e*) *Recurrence formulas* also exist: if there are any three functions $W_{k,m}(z)$ such that the corresponding values of k differ by integers, and the values of m also differ by integers, then the three functions are connected by a linear relation, the coefficients in which are polynomials in z.

6. *New classes of functions.*

As has been seen above, a number of functions which are really members of the family of functions $W_{k,m}(z)$ have in the past been separately discovered and investigated. There are other members of this family which have not hitherto been noticed, but which give promise of interesting properties. Among these may be mentioned the families of functions for which $m = 0$, and those for which $m = \frac{1}{2}$. With regard to the former of these classes, for example, it may be shown that if an arbitrary function $f(z)$ be expanded in the form

$$f(z) = a_0 W_{1/2,\,0}(z) + a_1 W_{3/2,\,0}(z) + a_2 W_{5/2,\,0}(z) + \cdots,$$

then the coefficients are given by the relation

$$a_n = \frac{1}{(n!)^2} \int_0^\infty f(z)\, W_{n+\frac{1}{2}}(z) \frac{dz}{z}.$$

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ON THE FACTORING OF LARGE NUMBERS.

BY PROFESSOR F. N. COLE.

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1. IN resolving a large number N into its prime factors, a table of quadratic remainders of N can be made to render efficient service in several different ways. For a twenty-two place N, the remainders of the table may be restricted to products of about seventy of the smallest prime numbers available. If, for example, N is always expressed in the form $N = x^2 \pm a$, with variable x and a, the remainders $\mp a$ will contain only those primes of which N is quadratic remainder. By a gradual elimination of common factors from the remainders $\mp a$, we finally obtain a table of remainders not admitting further reduction. Other forms for expressing N, such as $N = 2x^2 \pm b$, are of course often advantageous, according to circumstances. If the final table of remainders consists entirely of the individual primes employed, each with the proper sign $+$ or $-$, N is undoubtedly a prime number *; in fact a much smaller sequence of prime remainders would suffice to justify this conclusion, the wide range specified above being required only to ensure the success of the preliminary elimination process. If, on the other hand, it is found impossible, on repeated attempt,

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* For an interesting discussion of the use of the table of remainders see P. Seelhoff, *Zeitschrift für Mathematik und Physik*, vol. 31 (1886), pp. 166, 174, 306.