THE MODELS OF WHITTAKER AND KIRILLOV

GURBIR DHILLON

Abstract. We motivate why one would look for Whittaker and Kirillov models and state the basic theorem on their existence and uniqueness for GL_n.

1. Introduction

In the theory of representations, it can be conceptually helpful to treat representations as abstract vector spaces with a desired form of symmetry. On the other hand, it tends to be the case that to really get anywhere, you need a more concrete model. A beautiful and staggeringly successful strategy is to realize a family of representations as functions (or global sections of line bundles, D-modules, etc.) on a fixed space. The geometry of the space then teaches us many properties of the realized representations.

In the situation at hand, we are considering representations of $GL_n(k)$, where k is a nonarchimedean local field, e.g. $\mathbb{F}_q((t))$. The relevant concrete realizations will be the Whittaker and Kirillov models. To understand what is going on, we will first describe an analogous and simpler theory for $GL_n(\mathbb{C})$, and then in parallel illustrate how subtleties in adapting this for $GL_n(\mathbb{F}_q)$ and $GL_n(k)$ lead naturally to the Whittaker model.

1.1. **Intended audience.** These notes are geared towards readers who are comfortable with the representation theory of $GL_n(\mathbb{C})$, but have not spent much time around p-adic groups or finite groups of Lie type. We will focus mostly on heuristics and small examples which motivate the statements of theorems, and only prove things which we did not learn from a reference. We hope accordingly that the novice reader can enjoy this as a storybook, and leave with more context to approach the standard references.

2. $GL_n(\mathbb{C})$ and the universal principal series

In this section, we will replace $GL_n(\mathbb{C})$ with any connected, reductive group G over the complex numbers. The simple representations of G are classified by the 'theorem of the highest weight'. Recall that this means the following: if we fix a Borus $T \subset B \subset G$, then any simple representation L has a unique B stable line, and the eigenvalue of T on it determines L up to isomorphism. Let us tautologically rephrase this in a way which will motivate Whittaker's model.

To do so, we will use U, the unipotent radical of B. Recall that we have a functor Res_G^U which takes a G module and remembers only the action of U. This admits a right adjoint Ind_U^G. Finally, let me remind that in any finite length abelian category, such as U modules,

Date: Fall 2017.

one has the functor Cosoc, which takes an object to its maximal semisimple quotient, the *cosocle*.

Theorem 2.1. *Let L be a simple representation of $G(\mathbb{C})$. Consider the associated U modules:*
$$Q = \operatorname{Cosoc} \circ \operatorname{Res}_G^U L.$$
Then we have:

(i) Q is one dimensional, and there is a natural action of T on Q, which identifies Q with the lowest weight space of L.
(ii) There is a unique up to scaling nonzero map
$$L \to \operatorname{Ind}_U^G \mathbb{C}_0,$$
where $\mathbb{C}_0$ is the trivial representation of U.
(iii) Every simple G module shows up in $\operatorname{Ind}_U^G \mathbb{C}_0$ with multiplicity one.

Proof. We remind that, as with any cosocle, there is a canonical isomorphism
$$Q \simeq \bigoplus_V \operatorname{Hom}_U(\operatorname{Res}_G^U L, V)^\vee \otimes V,$$
where V runs over the isomorphism classes of simple modules of U. Since U is unipotent, the only simple module is $\mathbb{C}_0$. Accordingly, Q is simply the *coinvariants* of $\operatorname{Res}_G^U L$, i.e.
$$Q \simeq L/\langle ul - l \rangle, \qquad u \in U, l \in L.$$
Equivalently, writing $\mathfrak{u}$ for the Lie algebra of U, we have
$$Q \simeq L/\mathfrak{u}L.$$
Since U is normalized by T, $\mathfrak{u}L$ is preserved by the action of T. Given a nonzero vector of Q, take a nonzero weight component of it q. By the representation theory of $\mathfrak{sl}_2$, q must be lowest weight for each $\mathfrak{sl}_2$ triple associated to a simple root of G, i.e. a lowest weight vector, which proves (i).

Assertion (ii) now follows from (i) by adjunction. Also (iii) follows from (ii), using the complete reducibility of G modules. □

We have therefore seen the remarkable representation $\operatorname{Ind}_U^G \mathbb{C}_0$, sometimes called the *universal principal series*, stores every simple representation of G exactly once. Who is this remarkable creature? Let us recall that for any representation V of U, we have the explicit model of the induction:
$$\operatorname{Ind}_U^G V = \{f : G \to V : f(ug) = h \cdot f(h), u \in U, g \in G\}. \tag{2.2}$$
Here, by $f : G \to V$ we mean regular maps, i.e. $V^\vee \to \operatorname{Fun}(G)$, where $\operatorname{Fun}(G)$ denotes the algebra of polynomial functions of G. In the case of $V = \mathbb{C}_0$, the formula (2.2) is simply $\operatorname{Fun}(U\backslash G)$. I.e., the *base affine space* G/U knows the entire representation theory of G!

Let us conclude this section with two comments. First, G/U has a right action of T, with associated quotient G/B. Functions on G/U become sections of line bundles on G/B (the *Borel-Weil theorem*). As alluded to in the introduction, the geometry of G/B knows essentially everything about the category of G representations. For example, for a simple module L, the decomposition of $\mathrm{Res}^T_G L$ into eigenlines is given by the Weyl character formula. This formula can be fruitfully thought of as a 'linearization' of the Schubert stratification of G/B.

Second, one can also deduce the remarkable isotypic decomposition of $\mathrm{Fun}(G/U)$, and hence the rest of Theorem 2.1 from the Peter–Weyl theorem. This says that as a $G \times G$ module, we have

$$\mathrm{Fun}(G) \simeq \bigoplus_L L \boxtimes L^\vee,$$

where L ranges over the isomorphism classes of simple modules of G. It follows that

$$\mathrm{Fun}(G/U) \simeq \bigoplus_L L \boxtimes (L^\vee)^U,$$

where the upper U denotes the functor of invariants. By the highest weight theory, this is always one dimensional for the simple modules $L^\vee$, as desired.

3. A universal principal series for $G(\mathbb{F}_q)$ or $G(k)$?

Suppose we considering L, a simple complex representation of $G(\mathbb{F}_q)$ or $G(k)$, where k is a local field. Let us emphasize that e.g. in the case of $\mathbb{F}_q$ or $\mathbb{F}_q((t))$, we are crossing characteristics when talking about L. I.e., these are not at all algebraic representations. So, for $G(\mathbb{F}_q)$ we look at it as an abstract group, and for $G(k)$ we use its p-adic topology. The canonical example is $\mathbb{G}_a$, and $k = \mathbb{F}_q((t))$. In this case, $\mathbb{G}_a(k)$ is the additive group of Laurent polynomials, topologized so that high powers of t are small. We will work in the category of smooth representations of $G(k)$, i.e. representations V for which the action map $G \times V \to V$ is continuous, where a complex vector space V is given the *discrete* topology. Equivalently, we want the stabilizer of any point of V to be open in $G(k)$.

We can still make sense of $\mathrm{Ind}_U^G \mathbb{C}_0$. In the case of a finite group of Lie type, this will again be $\mathrm{Fun}(G(\mathbb{F}_q)/U(\mathbb{F}_q))$, i.e. $\mathbb{C}$ valued functions on the discrete coset space. In the case of a nonarchimedean local field k, this will be $\mathrm{Fun}^{sm}(G(k)/U(k))$, i.e. the subspace of smooth vectors in the space of all $\mathbb{C}$ set-theoretic functions $G(k)/U(k) \to \mathbb{C}$. To what extent does this function space continue to enjoy the beautiful properties we saw for $G(\mathbb{C})$?

It turns out two important things fail. First, not every simple representation shows up in this universal principal series $\mathrm{Ind}_U^G \mathbb{C}_0$. Second, those that show up can appear with multiplicity greater than one. Both of these properties can be seen explicitly in the example of $GL_2(\mathbb{F}_q)$. Indeed, let us recall the classification of simple $GL_2(\mathbb{F}_q)$ modules.[1] There are:

(1) $q-1$ determinant characters,
(2) $q-1$ twists of the Steinberg representation by a character,
(3) $\binom{q-1}{2}$ irreducible principal series representations, and
(4) $\binom{q}{2}$ *cuspidal* representations.

[1]This is a very useful example to internalize, if you have not done so already. Some friendly sources are Fulton and Harris [5, 5.2] and Bump [2, 4.1].

The definition of cuspidal, in the case of $GL_2(\mathbb{F}_q)$, amounts to having no maps to the universal principal series, which accounts for the first issue.[2] For the second issue, a direct calculation gives:

Proposition 3.1. *For any character χ of $B(\mathbb{F}_q)$, we have $\mathrm{Ind}_{B(\mathbb{F}_q)}^{GL_2(\mathbb{F}_q)} \chi$ has a two dimensional space of U invariants.*

Proof. A U invariant function in the induction is an $f : GL_2(\mathbb{F}_q) \to \mathbb{C}$ satisfying

$$f(ugb) = f(g)\chi(b), \qquad u \in U, b \in B, g \in G.$$

There are two Bruhat cells in $\mathbb{P}^1(\mathbb{F}_q)$, and each can support such a twisted function since $sUs \cap B = e$, where s denotes the nontrivial element of the Weyl group ('the open cell $Uw_\circ \to G/B$ is an embedding'). □

In particular, we see that the determinants and twisted Steinbergs show up in $\mathrm{Fun}(G/U)$ with multiplicity one, but the irreducible principal series show up with multiplicity two.

4. Twisting the universal principal series I: GL_2

We have seen that the universal principal series has some drawbacks. I.e., it fails to see some interesting representations, and sees others too many times. So, the most naive highest weight theory in this context fails! To fix this, we should go back to what led us to it for $G(\mathbb{C})$, i.e.

$$\mathrm{Cosoc} \circ \mathrm{Res}_G^U L.$$

For $U(\mathbb{C})$, the coinvariants were the same as the cosocle. But for $U(\mathbb{F}_q)$ or $U(k)$, there are *more* simple representations.

Example 4.1. *If we take U inside $GL_2(\mathbb{F}_p)$, then $U(\mathbb{F}_p) \simeq \mathbb{Z}/p\mathbb{Z}$, so its simple representations $\widehat{\mathbb{F}_p} \simeq \mu_p$, i.e. the generator acts by some p^{th} root of unity.*

For $U(\mathbb{F}_q)$ of G or semisimple rank greater than 1, e.g. $GL_3(\mathbb{F}_q)$, some of its representations are not characters, and presumably something similar holds for $U(k)$. *A priori,* we might need to look at these subtler representations. Before worrying about this, let us see whether in the simplest example $GL_2(\mathbb{F}_q)$ looking at a nontrivial character χ of U suffices, i.e. take 'twisted highest weight vectors'. So, let us look at the space of twisted functions:

$$\mathrm{Fun}^\chi(G/U) := \mathrm{Ind}_U^G \mathbb{C}_\chi,$$

and try sticking representations inside this. Which should χ should we pick? We will see in a second that the action of T on U permutes all its nontrivial characters, hence the choice will be irrelevant – the interested reader can check this now for $\mathbb{F}_p$.

To understand this gadget $\mathrm{Fun}^\chi(G/U)$, let us see what it does to our simplest representations, namely those arising from induction from the split torus, the types (1)-(3).

[2] Up to ignoring determinantal characters, the universal principal series is what one induces from a split torus, and the cuspidals are what one induces from a nonsplit torus. For more details on this remarkable story, look up Deligne–Lusztig theory. To our knowledge, the version of this story one loop up is still largely a mystery!

Lemma 4.2. *For any G module V, we have canonical isomorphisms:*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_U^G \mathbb{C}_\chi) \simeq J_\chi(V)^\vee,$$

where $J_\chi(V)$ denotes Jacquet's module *of twisted coinvariants:*

$$J_\chi(V) = V/\langle uv - \chi(u)v \rangle, \qquad u \in U, v \in V.$$

Proof. This is an immediate consequence of adjunction. Namely,

$$\mathrm{Hom}_G(V, \mathrm{Ind}_U^G \mathbb{C}_\chi) \simeq \mathrm{Hom}_U(\mathrm{Res}_G^U V, \mathbb{C}_\chi),$$

so we are picking out the χ isotypic component of $\mathrm{Cosoc} \circ \mathrm{Res}_G^U V$. □

Proposition 4.3. *For any character η of B, the space of intertwiners*

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G \eta, \mathrm{Ind}_U^G \chi)$$

is one dimensional.

Proof. By complete reducibility of representations of finite groups, $J_\chi(\mathrm{Ind}_B^G \eta)$ is explicitly the space of functions

$$f : G \to \mathbb{C} : f(bgu) = \eta(b) f(g) \chi(u), \qquad b \in B, u \in U.$$

Again, $B\backslash G/U$ is the Schubert stratification of $\mathbb{P}^1$. The open cell still supports an intertwiner, but the point orbit does not, since $\eta(U) = 1$, unlike $\chi(U)$. □

Corollary 4.4. *The characters (1) of $GL_2(\mathbb{F}_q)$ do not show up in $\mathrm{Fun}^\chi(G/U)$, and the twisted Steinbergs (2) and irreducible principal series representations (3) show up with multiplicity one.*

Okay, what about the representations of type (4), i.e. the cuspidals? To figure this out, we will use a result we will discuss later, namely $\mathrm{Fun}^\chi(G/U)$ *is multiplicity free*. Roughly, its endomorphism algebra will be the convolution algebra $\mathrm{Fun}^{\chi,\chi}(U\backslash G/U)$, and there will be a Gelfand trick to switch the two Us. In any case, $U\backslash G/U$ consists of $2(q-1)^2$ double cosets, namely $(q-1)^2$ from each Bruhat cell. Of these, a little arithmetic shows only $q-1$ from the small cell, coming from the central torus, and all $(q-1)^2$ from the big cell will support intertwiners. Using the $q-1$ to account for the twisted Steinbergs, and writing $(q-1)^2 = \binom{q}{2} + \binom{q-1}{2}$, we deduce *every cuspidal shows up in* $\mathrm{Fun}^\chi(G/U)$ *with multiplicity one.*

Summarizing, we have shown, modulo some facts to be given later, that for $GL_2(\mathbb{F}_q)$, $\mathrm{Fun}^\chi(G/U)$ breaks up as the sum of all the representations of types (2)-(4), with each showing up once. I.e., the twisted coinvariants J_χ of a simple module are always zero or one dimensional. So, we have recovered for $GL_2(\mathbb{F}_q)$ a fairly robust theory 'of the highest weight'.

With this, we might hope a similar twisted universal principal series $\mathrm{Fun}^\chi(G/U)$ for $GL_n(k), GL_n(\mathbb{F}_q)$, might give a highest weight theory. We will show this is the case, but we will need to pick a χ, and so we had better understand what the characters of U look like.

4.1. **Why coinvariants and not invariants?** Before moving on, the reader can reasonably object that nothing up until now has dictated that we use coinvariants rather than invariants. Of course, for a finite field, there is no difference. For k a local field, suppose we have a vector in a representation stable under U. By *smoothness*, it is stable under U and an open subgroup U' of $GL_2(k)$. Since U' is open, it must meet the other Bruhat cell of GL_2, and a little fiddling shows v must be invariant under all of $SL_2(F)$. In particular, if an irreducible representation has U invariants, then it is a determinantal character.[3]

5. Characters of $U(k)$ and $U(\mathbb{F}_q)$

As before, U denotes the unipotent radical of a Borel in G, a split reductive group. In this section, we only work things out for $U(K), K = \mathbb{F}_q$ or $\mathbb{F}_q((t))$; similar things hold in the number field setting. In our case, every element of $U(K)$ is torsion, so there is no difference between characters valued in S^1 and $\mathbb{C}^\times$. Let us write $\widehat{U(K)}$ for the space of all characters of $U(K)$. The natural action of $T(K)$ on $U(K)$ by conjugation induces an action on $\widehat{U(K)}$. This tells us that if L is a $G(K)$ module, $\mathrm{Res}^U_G L$ will have a T's worth of additional symmetry. This is analogous to the functor $\mathrm{Res}^{T(\mathbb{C})}_{G(\mathbb{C})}$, where the extra symmetry comes from Weyl group.

In any case, we would like to understand:

(1) What does $\widehat{U(K)}$ look like?
(2) What does $\widehat{U(K)}/T(K)$ look like?

For any Dynkin type, the first question has a simple answer provided one avoids some small characteristics. It turns out the second question can be rather subtle, so we only answer it for GL_n.

5.1. **Classifying the characters.** Let us answer question (1). To do so, recall that we have in good characteristics the root group parametrization of U:

$$U(K) = \prod_{\alpha>0} U_\alpha(K),$$

where $\alpha > 0$ denote the positive roots of G, and moreover

$$[U(K), U(K)] = \prod_{\alpha>0 \text{ non-simple}} U_\alpha(K).$$

It follows that characters of $U(K)$ are the same as characters of the abelian group

$$U(K)/[U(K), U(K)] = \prod_{\alpha \text{ simple}} \mathbb{G}_a(K).$$

For example, for GL_n, this is the usual business of only looking at matrix entries $e_{i,i+1}$, i.e. one above the diagonal. In any case, we have shown:

[3]There is a slightly subtlety here, since $GL_2(k)$ is not the product of the central torus and $SL_2(k)$, but this is nonetheless true.

Proposition 5.1. *There is a canonical isomorphism*

$$\widehat{U(K)} \simeq \prod_{\alpha \text{ simple}} \widehat{\mathbb{G}_a(K)}.$$

So, we are reduced to understanding the characters of $\mathbb{F}_q$ and $\mathbb{F}_q((t))$. Again, for the latter, we will only look at the smooth characters. First, as alluded to above, mapping to S^1 or $\mathbb{C}^\times$ are essentially a red herring in our case.

Lemma 5.2. *Pick any nontrivial character $\mathbb{F}_p \to \mathbb{C}^\times$. This induces isomorphisms:*

$$\iota_* : \mathrm{Hom}_{Ab}(\mathbb{G}_a(K), \mathbb{F}_p) \simeq \mathrm{Hom}_{Ab}(\mathbb{G}_a(K), \mathbb{C}^\times).$$

So, we need to classify smooth homomorphisms to $\mathbb{F}_p$. To do so, recall that there is a trace map $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$, which sends an element of $\mathbb{F}_q$ to the trace of multiplication by it, viewed as an endomorphism of the $\mathbb{F}_p$ vector space $\mathbb{F}_q$.

Proposition 5.3. *The natural map*

$$\mathbb{F}_q \to \mathrm{Hom}_{Ab}(\mathbb{F}_q, \mathbb{F}_p) : x \to \mathrm{tr}(x-),$$

is an isomorphism.

Proof. Since both sides are size q, this follows from the nondegeneracy of the trace pairing. □

To understand what $\widehat{U(K)}$ looks like for $K = \mathbb{F}_q((t))$, let us recall that there is a natural residue pairing:

$$\mathrm{Res} : \mathbb{F}_q((t)) \otimes \mathbb{F}_q((t))dt \to \mathbb{F}_q,$$

given by 'integration on the boundary circle of our punctured disk', i.e. the coefficient of t^{-1}. After tracing down to $\mathbb{F}_p$, this accounts for everything:

Proposition 5.4. *The natural map:*

$$\mathrm{tr} \circ \mathrm{Res} : \mathbb{F}_q((t))dt \to \mathrm{Hom}^{sm}_{Ab}(\mathbb{F}_q((t)), \mathbb{F}_p)$$

is an isomorphism.

Proof. Given an element of ϕ of $\mathrm{Hom}^{sm}_{Ab}(\mathbb{F}_q((t)), \mathbb{F}_p)$, let us restrict it to the subgroup $\mathbb{F}_q t^n \simeq \mathbb{F}_q$. By Proposition 5.3, we know this restriction is given by tracing against some unique $a_n \in \mathbb{F}_q$. Smoothness means that ϕ annihilates $t^n\mathbb{F}_q[[t]]$ for some $n \gg 0$, so ϕ is given by $\mathrm{tr} \circ \mathrm{Res}$ against

$$\phi(t)dt = \sum_{n \in \mathbb{Z}} a_n t^{-n-1} dt;$$

where by smoothness this is indeed a Laurent series. □

Let us summarize what happened. Up to an exponential, we thought about $\mathbb{F}_q$ just as an (abelian) Lie algebra $\mathfrak{h}$ over $\mathbb{F}_p$ equipped with a nondegenerate invariant bilinear form κ. When looping, if we write K_x for $\mathbb{F}_q((t))$ and ω_x for $\mathbb{F}_q((t))dt$, we used the pairing:

$$\langle X \otimes f, Y \otimes \alpha \rangle = \kappa(X, Y) \,\mathrm{Res}\, f\alpha, \qquad f \in K_x, \alpha \in \omega_x.$$

Up to taking a d, and replacing $\mathbb{F}_p$ with $\mathbb{C}$, this is essentially the cocycle defining Heisenberg and affine Lie algebras!

5.2. **Classifying $T(K)$ orbits.** Let $\alpha_i, i \in I$ be the simple roots of G. It is not hard to say what $t \in T(K)$ does to a character $(\psi_i)_{i\in I}$ of U. Namely, we replace $\psi_i(u)$ with $\psi_i(t^{\alpha_i}u)$.

For example, for SL_2, if we had a character:

$$\psi : \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \mapsto \psi(u),$$

then we have

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \cdot \psi : \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \mapsto \psi(t^2 u).$$

From the results in the previous section, if ψ is a nontrivial character, then all the other nontrivial characters can written uniquely as $\psi(tu)$, i.e. for SL_2 we run into the issue of not enough square roots in our field K. There are similar issues for other semisimple groups G.[4] For GL_2, by contrast, we can take

$$\begin{pmatrix} t & \\ & 1 \end{pmatrix} \psi(u) = \psi(tu).$$

This is the simplification afforded by working with GL_n alluded to earlier.

For general n, let $T'(K) \subset T(K)$ be the subgroup whose last diagonal entry is 1. Then, in appropriate coordinates, we have the action of $T'(K)$ on $\widehat{U(K)}$ is on rational points the usual diagonal action

$$\mathbb{G}_m^{n-1} \times \mathbb{A}^{n-1} \to \mathbb{A}^{n-1}.$$

In particular, there is no different between $T'(K)$ and $T(K)$ orbits (aside from stabilizers), and they are indexed by subsets of $1, \ldots, n-1$, i.e. how many coordinates of a vector in K^{n-1} are nonzero. We deduce:

Proposition 5.5. *For $G = GL_n$, there is a canonical bijection:*

$$\widehat{U(K)}/T(K) \simeq \{ \text{ subsets of the simple roots } \}.$$

Let us call a point of the open orbit, i.e. all coordinates nonzero, *generic*. I.e., if we pick a nontrivial character ψ of $\mathbb{G}_a(K)$, the generic characters are the orbit of

$$\begin{pmatrix} 1 & u_{1,2} & * & * & * \\ & 1 & u_{2,3} & * & * \\ & & \ddots & \ddots & * \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix} \to \psi(u_{1,2} + u_{2,3} + \ldots u_{n-1,n}).$$

[4]There won't be an issue for some of them, i.e. G_2, F_4, E_8, whose simply connected and adjoint forms coincide.

Here are some comments. First, if we are working with $G(\mathbb{C})$, for any connected reductive group, we have all the roots we like. So, Proposition 5.5 applies to the characters of $U(\mathbb{C})$, i.e. $T(\mathbb{C})$ orbits are in canonical bijection with parabolics up to conjugacy. Second, a similar classification, for almost the same reasons, shows up when classifying singular oper connections on a (punctured) formal disk.

6. Twisted universal principal series II: GL_n

In the following statements, we continue for the nonarchimedean case to work in the category of smooth, admissible representaions.

Let us phrase the results of Section 4.1 in terms of 5.2. Namely, we saw that the naive universal principal series $\mathrm{Fun}(G/U)$ failed to have two properties we enjoyed with $G(\mathbb{C})$ - containing most of the interesting representations of GL_2 and being multiplicity free. There are two orbits of $T(K)$ on $\widehat{U(K)}$, namely trivial and nontrivial characters. When we twisted by a nontrivial character, $\mathrm{Fun}^\chi(G/U)$ had the desired properties, trading seeing the determinantal characters for seeing cuspidals.

For GL_n, we will again take the most nondegenerate orbit, i.e. let us look at $\mathrm{Fun}^\chi(G/U) = \mathrm{Ind}_U^G \mathbb{C}_\chi$, where χ is generic. The big theorem is:

Theorem 6.1. *(Multiplicity one) For χ a generic character, and L a simple $G(K)$ module, we have:*

(1) $\mathrm{Hom}(L, \mathrm{Ind}_U^G \mathbb{C}_\chi)$ *is at most one dimensional.*
(2) If L is cuspidal, then $\mathrm{Hom}(L, \mathrm{Ind}_U^G \mathbb{C}_\chi)$ *is one dimensional.*

Let us tautologically rephrase Theorem 6.1. Namely, for a G module V, we have as in the case of GL_2 the Jacquet module of twisted coinvariants:

$$J_\chi(V) = V/\langle uv - \chi(u)v\rangle, \qquad u \in U, v \in V.$$

For the same reasons as for GL_2, we have natural isomorphisms

$$J_\chi(V)^\vee \simeq \mathrm{Hom}_G(V, \mathrm{Ind}_U^G \mathbb{C}_\chi).$$

Corollary 6.2. *For any simple $G(K)$ module L, the Jacquet module $J_\chi(L)$ is at most one dimensional, and is one dimensional if L is cuspidal.*

An element of $J_\chi(V)^\vee$ is called a Whittaker function, i.e. a functional

$$f : V \to \mathbb{C} \qquad f(uv) = \chi(u)v \qquad u \in U, v \in V.$$

The image of the associated morphism $V \to \mathrm{Fun}^\chi(G/U)$ is called the Whittaker model.

As to proofs of Theorem 6.1, I have nothing to say other than the standard proofs. So, for $GL_n(\mathbb{F}_q)$ a friendly reference is the notes of Bump [3, § 4-6], and for $GL_n(k)$, for K a local field, I do not know a better reference than the original (legendary!) paper of Bernstein–Zelevinsky [1, Ch. 3]. For just GL_2, one can also see the book of Bump [2, Ch. 4]. In reading them, it may be helpful to start with $GL_n(\mathbb{F}_q)$. This often contains many representative ideas from proofs for $GL_n(k)$, but has the significant simplification that one has to deal with functions rather than distributions, e.g. already for the characters of representations.

6.1. **Whittaker models elsewhere in representation theory.** As an aside, let us mentioned some related interesting phenomena.

6.1.1. $G(\mathbb{C})$. For a connected reductive group $G(\mathbb{C})$, the only simple representation of $U(\mathbb{C})$ is the trivial module. However, the Lie algebra $\mathfrak{u}(\mathbb{C})$ has many nonintegrable characters, which again are classified up to T conjugacy by subsets of the simple roots. There is a notion of Whittaker function for a $\mathfrak{g} = Lie(G)$ module, and these were pioneered in the generic case by Kostant [6]. In geometry, they correspond to certain irregular holonomic D-modules on the flag variety [7]. As one degenerates the character χ to 0, these relate (nearby cycles) in an interesting way to the usual Category $\mathcal{O}$ in a manner which is not yet fully understood.

6.1.2. $G(\mathbb{C}((z)))$ *and beyond.* For a loop group over the complex numbers, you should expect there is a similar notion of Whittaker model to the local field case k discussed here. Naively, $\mathrm{Fun}^{\chi}(G(k)/U(k))$ should correspond to twisted D-modules on $G((z))/U((z))$. However, making sense of this kind of gadget is famously subtle, see e.g. [8].

To compare with Kostant's notion, we are using a 'semi-infinite' Borel and working one categorical level up. For the naive Whittaker model with respect to the Iwahori subgroup, there has been some work done in recent years [4]. It seems reasonable to wish there should be a similar story for any Coxeter group in the spirit of Soergel's modules.

7. Kirillov's model: GL_2

Finally, we should mention another model for representations for GL_n, which to my knowledge does not have a reasonable analogue for groups outside type A. Let us start with $GL_2(\mathbb{F}_q)$.

7.1. **Finite fields.** We saw that the cuspidals for $GL_2(\mathbb{F}_q)$ all showed up in $\mathrm{Fun}^{\chi}(G/U)$ once, and in $\mathrm{Fun}(G/U)$ zero times. We saw the twisted Steinbergs show up in both with multiplicity one. Finally, we saw that the simple principal series showed up in $\mathrm{Fun}(G/U)$ twice, and in $\mathrm{Fun}^{\chi}(G/U)$ once.

Unwinding, we are saying that (i) if L is cuspidal, we have

$$\mathrm{Res}_G^U L \simeq \bigoplus_{\chi \neq 0} \mathbb{C}_{\chi}.$$

Similarly, if (ii) L is a twisted Steinberg, we have

$$\mathrm{Res}_G^U L \simeq \mathbb{C}_0 \oplus \bigoplus_{\chi \neq 0} \mathbb{C}_{\chi},$$

and if (iii) L is a simple principal series, we have

$$\mathrm{Res}_G^U L \simeq \mathbb{C}_0^{\oplus 2} \oplus \bigoplus_{\chi \neq 0} \mathbb{C}_{\chi}.$$

This can be thought of as a strengthening of the statement that cuspidals have dimension $q-1$, twisted Steinbergs have dimension q, and simple principal series have dimension $q+1$.

How can we see this $q-1$ dimensional subspace that they all have 'in common'? Let us recall the smaller torus $T' = \begin{pmatrix} t & \\ & 1 \end{pmatrix}$ which permuted the nontrivial characters of χ. If we form the subgroup $M = T'U$ of GL_2, the so-called *mirabolic*, then for the usual reasons:

$$\mathrm{Res}^U_M \mathrm{Ind}^M_U \mathbb{C}_\eta \simeq \bigoplus_{\chi \neq 0} \mathbb{C}_\chi,$$

for any nonzero η. The space $\mathrm{Ind}^M_U \mathbb{C}_\eta$ is called Kirillov's module.

In what way does Kirillov's module realize the $q-1$ dimensional subspace all these have in common? Well, we should produce a map, and we unwind:

$$\mathrm{Hom}_M(\mathrm{Res}^M_G L, \mathrm{Ind}^M_U \mathbb{C}_\eta) \simeq \mathrm{Hom}_U(\mathrm{Res}^U_M \mathrm{Res}^M_G L, \mathbb{C}_\eta) \simeq \mathrm{Hom}_U(\mathrm{Res}^U_G L, \mathbb{C}_\eta) \simeq J_\eta(L)^\vee,$$

so by our previous results, the space of intertwiners is one dimensional. In all cases, the map is surjective, and has kernel the trivial isotypic component of the module, i.e. is 0, 1, or 2 dimensional as in the decompositions deduced above.

In particular, for L cuspidal, we have $\mathrm{Res}^M_G L \simeq \mathrm{Ind}^M_U \mathbb{C}_\eta$, and this isomorphism is known as *Kirillov's model.* We should also mention:

Proposition 7.1. *Kirillov's model is a simple M representation.*

7.2. **Local fields.** One can mimick the definitions and arguments from above. First, we should note that similarly to $GL_2(\mathbb{F}_q)$, we have:

Proposition 7.2. *Every simple module L of $GL_2(k)$ which is not a character embeds (uniquely) into $\mathrm{Ind}^G_U \mathbb{C}_\chi$, i.e. has a Whittaker model.*

Unlike $GL_2(\mathbb{F}_q)$, the actual coinvariants $J_0(L)$ cannot be split back into L as a U module i.e. the U invariants of L vanish when L is a non-character. This implies, with a little work, that:

Proposition 7.3. *For a simple module L which is not a character, the canonical up to scalars map*

$$\mathrm{Res}^M_G L \to \mathrm{Ind}^M_U \mathbb{C}_\chi$$

is an embedding. When L is cuspidal, the image is always the compactly supported twisted funtions in $\mathrm{Fun}^\chi(M/U)$.

By the way, the numbers 0, 1, 2 for cuspidals, twisted Steinbergs, and simple principal series *do* show up here too. Namely, if we think of $\mathrm{Ind}^M_U \mathbb{C}_\chi$ as $\mathrm{Fun}^\chi(M/U)$, the twisted functions of compact support are of codimension 0, 1, 2 in the image of L in these three cases, respectively.

8. Kirillov's model: GL_n

Let us outline how this extends to GL_n, with proofs and more discussion to be found in the same references discussed earlier. The analogue of T' for GL_2 is still called the mirabolic M for GL_n. It consists of matrices whose last row consists of zeroes, safe for the bottom right corner, which is a one. In particular, its unipotent radical is roughly the last column, and reductive quotient is GL_{n-1}. This *absolutely* should be compared to restricting from S_n to S_{n-1}, but padding with the n^{th} Jucy–Murphys operator.

In any case, for a generic character χ of U, we can produce Kirillov's module $\mathrm{Ind}_U^M \mathbb{C}_\chi$.

8.1. **Finite fields.** In this case, we have:

Theorem 8.1. *Kirillov's module is simple. For a simple G module L with a Whittaker model, the associated map of M modules*

$$M \to \mathrm{Ind}_U^M \mathbb{C}_\chi$$

is surjective, and if L is cuspidal is an isomorphism.

8.2. **Local fields.** In this case, we have:

Theorem 8.2. *For a simple G module L witha Whittaker model, the associated map of M modules*

$$M \to \mathrm{Ind}_U^M \mathbb{C}_\chi$$

is an embedding. Moreover, if L is cuspidal, the image is always the compactly supported twisted functions in $\mathrm{Fun}^\chi(M/U)$.

References

[1] Joseph Bernstein and Andrei Zelevinsky. Representations of the group GL(n,F), where F is a non-archimedean local field.
[2] Daniel Bump. *Automorphic forms and representations.*
[3] Daniel Bump. Notes on representations of GL(r) over a finite field.
[4] Rencai Lü Dražen Adamović and Kaiming Zhao. Whittaker modules for the affine lie algebra $A_1^{(1)}$.
[5] William Fulton and Joe Harris. *Representation theory a first course.*
[6] Bertram Kostant. On Whittaker vectors and representation theory.
[7] Dragan Miličić and Wolfgang Soergel. Twisted Harish-Chandra sheaves and Whittaker modules: the nondegenerate case.
[8] Sam Raskin. Chiral principal series categories ii: the factoriable Whittaker category.