Cauchy-Riemann Equations Write a complex-differentiable function $f(z)$ as $f(x,y) = u(x,y) + iv(x,y)$. Then

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

Let the complex variable h moves along the real axis, i.e., $h = t \in \mathbb{R}$, then $f'(z)$ is equal to

$$\lim_{t\to 0} \frac{f(x+t,y) - f(x,y)}{t} = \lim_{t\to 0} \left(\frac{u(x+t,y) - u(x,y)}{t} + i \frac{v(x+t,y) - v(x,y)}{t} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

On the other hand, let the complex variable h moves along the imaginary axis, i.e., $h = it$ where $t \in \mathbb{R}$, then $f'(z)$ is also equal to

$$\lim_{t\to 0} \frac{f(x,y+t) - f(x,y)}{it} = \lim_{t\to 0} \left(\frac{u(x,y+t) - u(x,y)}{it} + i \frac{v(x,y+t) - v(x,y)}{it} \right) = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}.$$

$\Longrightarrow$

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y},$$

so that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which are called the *Cauchy-Riemann Equations.*

Proposition 3.1.2 *Let $f = u + iv$ be a complex-valued function defined on a domain $D \subset \mathbb{C}$ where u and v are C^1-smooth real-valued functions [1]. Then*

$$f \text{ is holomorphic} \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ hold on } D.$$

Proof: (Optional reading) ($\Longrightarrow$) It has been proved.

($\Longleftarrow$) For any fixed $z \in D$, to show:

$$\frac{f(z+s+it) - f(z)}{s+it} \longrightarrow u_x(z) + iv_x(z), \quad \text{as } s+it \to 0. \tag{3.1}$$

[1] By C^1-smooth, we mean that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous on D. The C^1-smoothness condition can be dropped by a theorem of Goursat which will be proved later. We also remark that in some books, the C^1-smoothness condition is added to the definition of holomorphic function.

Here we denote $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$. In fact, the left term in (3.1) is

$$\frac{f(z+s+it)-f(z)}{s+it} = \frac{u(x+s,y+t)+iv(x+s,y+t)-u(x,y)-iv(x,y)}{s+it}$$
$$= \frac{u(x+s,y+t)-u(x,y)}{s+it} + i\frac{v(x+s,y+t)-v(x,y)}{s+it}$$
$$= \frac{\big[u(x+s,y+t)-u(x,y+t)\big]+\big[u(x,y+t)-u(x,y)\big]}{s+it}$$
$$+i\frac{\big[v(x+s,y+t)-v(x,y+t)\big]+\big[v(x,y+t)-v(x,y)\big]}{s+it}$$
$$= \frac{u_x(x+s_1,y+t)s+u_y(x,y+t_1)t}{s+it} + i\frac{v_x(x+s_2,y+t)s+u_y(x,y+t_2)t}{s+it}$$

where $|s_1| \leq |s|$ and $|t_1| \leq |t|$. Here the last equality holds because of the C^1-smoothness condition so that we can apply the Lagrange Mean Theorem. [2]

The right term in (3.1) is

$$u_x(z)+iv_x(z) = \frac{(s+it)(u_x(z)+iv_x(z))}{s+it} = \frac{su_x(z)-tv_x(z)}{s+it} + i\frac{tu_x(z)+sv_x(z)}{s+it}.$$

In order to finish the proof of (3.1), we only prove the real part (the proof for the imaginary part is similar) in above two formulas, i.e., we need to prove:

$$\frac{u_x(x+s_1,y+t)s+u_y(x,y+t_1)t}{s+it} \;\to\; \frac{su_x(z)-tv_x(z)}{s+it}, \quad as\ s+it\to 0,$$

i.e., to prove

$$\frac{u_x(x+s_1,y+t)s+u_y(x,y+t_1)t}{s+it} - \frac{su_x(x,y)-tv_x(x,y)}{s+it} \;\to\; 0, \quad as\ s+it\to 0,$$

i.e., to prove

$$\frac{s[u_x(x+s_1,y+t)-u_x(x,y)]}{s+it} + \frac{t[u_y(x,y+t_1)+v_x(x,y)]}{s+it} \;\to\; 0, \quad as\ s+it\to 0,$$

By Cauchy-Riemann Equations, we replace v_x above by $-u_y$, we need to prove:

$$\frac{s}{s+it}[u_x(x+s_1,y+t)-u_x(x,y)] + \frac{t}{s+it}[u_y(x,y+t_1)-u_y(x,y)] \;\to\; 0 \tag{3.2}$$

[2]The Lagrange Mean Value Theorem: If f is a C^1-smooth function, then $f(b)-f(a)=f'(c)(b-a)$ for some point $c \in (a,b)$.

as $s + it \to 0$.

Now we have

$$\left|\frac{s}{s+it}\right| = \frac{|s|}{\sqrt{s^2+t^2}} \le 1, \quad \left|\frac{t}{s+it}\right| = \frac{|t|}{\sqrt{s^2+t^2}} \le 1,$$

and $u_x(x+s_1, y+t) - u_x(x,y) \to 0$, $u_y(x, y+t_1) - u_y(x,y) \to 0$, as $s + it \to 0$, because of the C^1-smoothness condition. Then (3.2), as well as (3.1), is proved. □

Remarks on Holomorphic Functions In 1752, with the problem of hydrodynamics, D'Alembert [3] first established the system of differential equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

for a function $f(x+iy) = u + iv$ of a complex variable $z = x + iy$. In 1761, D'Alembert also showed that u and v satisfy the equation (today we call it the Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. [4]

Euler also had written down these equations in 1752 in connection with his work on the motion of fluids. In papers presented to the St. Petersburg Academy in 1777 and 1778, Euler consider integral $\int Z dz$ where $z = x + y\sqrt{-1}$ and $Z = M + N\sqrt{-1}$ with the condition

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}, \ \frac{\partial N}{\partial x} = -\frac{\partial M}{\partial y}.$$ [5]

In spite of the important works of d'Alembert's and Euler's results, complex analysis emerged as a proper domain of modern mathematics only during the 19th century as a result of the work of Augustin Louis Cauchy, Bernhard Riemann and Karl Weierstrass.

In 1814, Cauchy obtain the "Cauchy-Riemann equations" and complex integrals.

Although Cauchy is the founder of the theory of complex analysis, he did not make a clear definition of holomorphic functions. In his masterpiece 1825 paper which contains Cauchy Integral Theorem and in his 1831 paper which contains Cauchy's Integral Formula, Cauchy only put the " finite and continuous" condition on the functions. Cauchy may took it for granted that a continuous function satisfies the Cauchy-Riemann equations. [6] In fact

[3]Jean le Rond d'Alembert (1717-1783) was a French mathematician, mechanician, physicist and philosopher. D'Alembert's method for the wave equation is named after him.

[4]Hans Niels Jahnke (editor), *A History of Analysis*, AMS, 2003, p.213.

[5]Hans Niels Jahnke (editor), *A History of Analysis*, AMS, 2003, p.214.

[6]Hans Niels Jahnke (editor), *A History of Analysis*, AMS, 2003, p.220-221 and 225.

Remmert said[7]: "Actually, as late as 1851 Cauchy still had no exact definition of the class of functions for which his theory was valid."

Riemann's definition of a holomorphic function $f = u + iv$ is that at a point and in its neighborhood if $f(z)$ is continuous and differentiable and satisfies what we now call the Cauchy-Riemann equations. Riemann was the first to require that the existence of the derivatives dw/dz means that the limit of $\Delta w/\Delta z$ must be the same for every approach of $z + \Delta z$ to z. [8]

Weierstrass rejected Cauchy's and Riemann's definition of a holomorphic functions. His "power series" definition (i.e., the definition of analytic functions) is an arithmetic approach.

For a long time, Riemann's construction of complex function theory (beginning with complex differentiation) and Weierstrass' construction (beginning with power series) are two different approaches, which influenced textbooks on complex analysis until the early decades of the 20th century.

When the theory of functions reached maturity, all of these definitions were all seen to be equivalent. Most of many terms (for examples, "regular", "synectic" and "monogenic") have now been forgotten.

The word "holomorphic" was in troduced in 1875 in a book (the 2nd edition) by Briot and Bouquet. In their first edition, they used instead of "holomorphic" the designation "synectic," which goes back to Cauchy. [9]

Another Version of the Cauchy-Riemann Equations We define differential operators:

$$\partial_z f = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right),\ \overline{\partial}_z f = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right),$$

Proposition 3.1.3 *Let $D \subset \mathbb{C}$ be a domain. Then*

(i) $\overline{\partial}_z f = 0$ holds on $D \iff f\ satisfies\ the\ Cauchy-Riemann\ conditions: \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ on D.

(ii) $\partial_z f = f' = u_x + iv_x = u_y + iv_y$ for any holomorphic function f defined on D.

[7] R. Remmert, *Theorey of Complex Functions*, GTM 122, Springer, 1991, p.62.

[8] Morris Kline, *Mathematical Thought from Ancient to Modern Times*, volume 1, New York Oxford, Oxford University Press, II, 1972, p.658.

[9] R. Remmert, *Theorey of Complex Functions*, GTM 122, Springer, 1991, p.61.

Proof: (i) $\overline{\partial}_z f = 0 \Longleftrightarrow \frac{u+iv}{\partial x} - \frac{1}{i}\frac{u+iv}{\partial y} = 0, \Longleftrightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 0, \Longleftrightarrow$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \quad and \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0,$$

which are the Cauchy-Riemann Equations.

(ii) $\partial_z f = \frac{1}{2}\left(\frac{\partial(u+iv)}{\partial x} + \frac{1}{i}\frac{\partial(u+iv)}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$. Applying the Cauchy-Riemann Equations, we obtain

$$\partial_z f = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = f'.$$

Here for the last equality we used the definition of holomorphic function. □

[**Example**] (1) Let $f(z) = 2z^2 + 3z - 6$. Then since $\partial_{\overline{z}} f = 0$, f is holomorphic and hence $f'(z) = \partial_z f = 4z + 3$.

(2) Let $f(z) = 2z^2 + 3\overline{z} - 4z + 1$. Since $\partial_{\overline{z}} f = \frac{\partial}{\partial \overline{z}}(2z^2 + 3\overline{z} - 4z + 1) = 3$, f is not a holomorphic function so that f' does not exit.

In general, if a power series $f(z)$ does not contain $\overline{z}^k$ terms, f is holomorphic because $\partial_{\overline{z}} f = 0$ (The Cauchy-Riemann equations); if a power series or a polynomial f contains $\overline{z}^k$ terms, it is not holomorphic. □