# Partial Fractions

Combining fractions over a common denominator is a familiar operation from algebra:

$$\frac{2}{x-3} + \frac{3}{x^2+1} = \frac{2x^2+3x-7}{x^3-3x^2+x-3} \tag{1}$$

From the standpoint of integration, the left side of Equation 1 would be much easier to work with than the right side. In particular,

$$\begin{aligned} \int \frac{2}{x-3} + \frac{3}{x^2+1}\,dx &= \int \frac{2}{x-3}\,dx + \int \frac{3}{x^2+1}\,dx \\ &= 2\ln|x-3| + 3\arctan x + C \end{aligned}$$

So, when integrating rational functions it would be helpful if we could *undo* the simplification going from left to right in Equation 1. Reversing this process is referred to as finding the *partial fraction decomposition* of a rational function.

## Getting Started

The method for computing partial fraction decompositions applies to all rational functions with one qualification:

> **The degree of the numerator must be less than the degree of the denominator.**

One can always arrange this by using polynomial long division, as we shall see in the examples.

Looking at the example above (in Equation 1), the denominator of the right side is $x^3 - 3x^2 + x - 3 = (x-3)(x^2+1)$. Factoring the denominator of a rational function is the first step in computing its partial fraction decomposition. Note, the factoring must be complete (over the real numbers). In particular this means that each individual factor must either be linear (of the form $ax+b$) or irreducible quadratic (of the form $ax^2+bx+c$).

When is a quadratic polynomial irreducible? If a quadratic polynomial factors, such as $x^2 - x - 6 = (x-3)(x+2)$, then it has at least one root. Similarly, if it has a root $r$, then it must have a factor of $x - r$. Thus, a quadratic polynomial is irreducible iff it has no real roots. This is easy to determine using the quadratic formula: the roots of $ax^2+bx+c$ are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

and these are real numbers iff $b^2 - 4ac \geq 0$. Thus, this quadratic polynomial is irreducible iff its discriminant $b^2 - 4ac < 0$.

## Finding the right form

An important step in this process is to know the right form of the decomposition. It will be a sum of terms in which the numerators contain coefficients (such as the $A$, $B$, and $C$ above). In fact, the number of these unknown coefficients will always be equal to the degree of the denominator.

After the denominator is factored and like terms are collected, we can use the following rules to determine the terms in the decomposition.

- For a linear term $ax + b$ we get a contribution of $\dfrac{A}{ax+b}$.
- For a repeated linear term, such as $(ax+b)^3$, we get a contribution of

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}.$$

  We have *three* terms which matches that $(ax + b)$ occurs to the *third* power.

- For a quadratic term $ax^2 + bx + c$ we get a contribution of $\dfrac{Ax+B}{ax^2+bx+c}$.
- For a repeated quadratic term such as $(ax^2 + bx + c)^2$ we get a contribution of

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}.$$

These rules can be mixed together in any way.

Here we give several rational functions and the form of their partial fraction decompositions.

**Example 1.**

$$\frac{2x^2+1}{x^3-x^2-8x+12} = \frac{2x^2+1}{(x-2)^2(x+3)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+3}$$

**Example 2.**

$$\frac{x^2+1}{(x^2+x+2)(x+7)} = \frac{Ax+B}{x^2+x+2} + \frac{C}{x+7}$$

**Example 3.**

$$\frac{1}{(x^2+2x+5)^2(x-1)(x+2)} = \frac{Ax+B}{x^2+2x+5} + \frac{Cx+D}{(x^2+2x+5)^2} + \frac{E}{x-1} + \frac{F}{x+2}$$

**Example 4.**

$$\frac{1}{(x^2-3)^2} = \frac{1}{(x-\sqrt{3})^2(x+\sqrt{3})^2} = \frac{A}{x-\sqrt{3}} + \frac{B}{(x-\sqrt{3})^2} + \frac{C}{x+\sqrt{3}} + \frac{D}{(x+\sqrt{3})^2}$$

In the last example we needed to factor the denominator further.

## Computing the coefficients

Once we have determined the right form for the partial fraction decomposition of a rational function, we need to compute the unknown coefficients $A$, $B$, $C$, .... There are basically two methods to choose from for this purpose. We will now look at both methods for the decomposition of

$$\frac{2x-1}{(x+2)^2(x-3)}.$$

By the rules above, its partial fraction decomposition takes the form

$$\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x-3}.$$

Setting these equal and multiplying by the common denominator gives

$$2x-1 = A(x+2)(x-3)+B(x-3)+C(x+2)^2. \tag{2}$$

Our first method is to substitute different values for $x$ into Equation 2 and deduce the values of $A$, $B$, and $C$. It helps to start with values of $x$ which are roots of the original denominator since they will make some of the terms on the right side vanish.

- Using $x = 3$ gives $2(3) - 1 = 0 + 0 + C \cdot 5^2$. Thus, $C = 1/5$.
- From $x = -2$, we learn that $-5 = 0 + B(-5) + 0$, and so $B = 1$.
- We have run out of roots of the denominator, and so we pick a simple value of $x$ to finish off. From $x = 0$ we find $-1 = -6A - 3B + 4C$. Using our values for $B$ and $C$, this becomes $-1 = -6A - 3(1) + 4(1/5)$ and so $A = -1/5$.

Therefore,

$$\frac{2x-1}{(x+2)^2(x-3)} = \frac{-1/5}{x+2}+\frac{1}{(x+2)^2}+\frac{1/5}{x-3}.$$

The second method is used in the textbook (pp. 371–372). After setting up the decomposition, again we clear denominators to produce Equation 2. However, this time we will expand the right side and collect like terms:

$$\begin{aligned} 2x-1 &= Ax^2 - Ax - 6A + Bx - 3B + Cx^2 + 4Cx + 4C \\ &= (A+C)x^2 + (-A+B+4C)x + (-6A-3B+4C) \end{aligned}$$

For these polynomials to be equal, their coefficients must be equal, leading us to the system of equations:

$$\begin{aligned} 0 &= A + C && \text{from the } x^2 \text{ terms} \\ 2 &= -A + B + 4C && \text{from the } x \text{ terms} \\ -1 &= -6A - 3B + 4C && \text{from the constant terms} \end{aligned}$$

We now have to solve these three equations with three unknowns. You may use any standard method for solving the systems of equations. Here, we will use substitution.

From the first equation, $C = -A$. Substituting into the other equation yields

$$\begin{aligned} 2 &= -A + B + 4(-A) = -5A + B \\ -1 &= -6A - 3B + 4(-A) = -10A - 3B \end{aligned}$$

Solving the first equation for $B$ gives $B = 5A + 2$. Substituting this into the second equation yields

$$-1 = -10A - 3(5A + 2) = -25A - 6$$

so $-25A = 5$, or $A = -1/5$. Then $B = 5A + 2$ gives $B = 1$ and $C = -A$ gives $C = 1/5$.

One advantage of this method is that it *proves* that the given decomposition is correct. By contrast, the first method of determining the coefficients *assumes* that we have set up the decomposition correctly.

## Examples

Here we use partial fractions to compute several integrals:

**Example 5.** $\displaystyle\int \frac{x+3}{(x^2-1)(x+5)}\,dx$

*Solution:* Factoring the denominator completely yields $(x-1)(x+1)(x+5)$, and so

$$\frac{x+3}{(x^2-1)(x+5)} = \frac{x+3}{(x-1)(x+1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+5}$$

Clearing denominators gives the equation:

$$x + 3 = A(x+1)(x+5) + B(x-1)(x+5) + C(x-1)(x+1)$$

Since the denominator has distinct roots, the quickest way to find $A$, $B$, and $C$ will be to plug in the roots of the original denominator:

- $x = 1$ gives $4 = 12A \implies A = 1/3$
- $x = -1$ gives $2 = -8B \implies B = -1/4$
- $x = -5$ gives $-2 = -24C \implies C = 1/12$

Putting it all together, we find

$$\begin{aligned} \int \frac{x+3}{(x^2-1)(x+5)}\,dx &= \int \frac{1/3}{x-1} + \frac{-1/4}{x+1} + \frac{1/12}{x+5}\,dx \\ &= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{12}\int \frac{dx}{x+5} \\ &= \frac{1}{3}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{12}\ln|x+5| + C \end{aligned}$$

Note, we use $C$ here for the constant of integration even though $C$ has occured earlier in the problem as a coefficient. However, it is unlikely that confusion will arise by re-using $C$ in this way.

**Example 6.** $\int \frac{2x-2}{(x^2+x+4)(x+2)}\,dx$

*Solution:* We first check that the quadratic factor is irreducible by computing its discriminant: $1^2 - 4 \cdot 1 \cdot 4 = -15 < 0$. Thus, the denominator is already factored completely and we are ready to set up the partial fractions:

$$\frac{2x-2}{(x^2+x+4)(x+2)} = \frac{Ax+B}{x^2+x+4} + \frac{C}{x+2}$$

Clearing denominators leads to the equation:

$$2x - 2 = (Ax+B)(x+2) + C(x^2+x+4) \tag{3}$$

Evaluating both sides at $x = -2$ gives one coefficient:

$$-6 = (-2A+B)(0) + C(6) \quad \Longrightarrow \quad C = -1$$

Next, we try $x = 0$ in Equation 3:

$$-2 = (0A+B)(2) + (-1)(4) \quad \Longrightarrow \quad -2 = 2B - 4 \quad \Longrightarrow \quad B = 1$$

Finally, we use another simple value of $x$ in Equation 3, namely $x = 1$:

$$0 = (A+1)(3) + (-1)(6) \quad \Longrightarrow \quad 0 = 3A - 3 \quad \Longrightarrow \quad A = 1$$

Thus,

$$\begin{aligned}
\int \frac{2x-2}{(x^2+x+4)(x+2)}\,dx &= \int \frac{x+1}{x^2+x+4} + \frac{-1}{x+2}\,dx \\
&= \int \frac{x+1}{x^2+x+4}\,dx - \int \frac{dx}{x+2} \\
&= \int \frac{x+1}{x^2+x+4}\,dx - \ln|x+2|
\end{aligned}$$

To integrate $\int \frac{x+1}{x^2+x+4}\,dx$, we complete the square $x^2+x+4 = (x+\frac{1}{2})^2 + \frac{15}{4}$, and make a substitution $u = x + \frac{1}{2}$ (so $u - \frac{1}{2}$ and $du = dx$) to get

$$\begin{aligned}
\int \frac{x+1}{x^2+x+4}\,dx &= \int \frac{x+1}{(x+\frac{1}{2})^2 + \frac{15}{4}}\,dx \\
&= \int \frac{(u-\frac{1}{2})+1}{u^2+\frac{15}{4}}\,du \\
&= \int \frac{u+\frac{1}{2}}{u^2+\frac{15}{4}}\,du
\end{aligned}$$

From a table of integrals, we can now evaluate this to be

$$\begin{aligned}
&= \frac{1}{2}\ln\left|u^2+\frac{15}{4}\right| + \frac{1/2}{\sqrt{15/4}}\arctan\frac{u}{\sqrt{15/4}} + C \\
&= \frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)^2+\frac{15}{4}\right| + \frac{1}{\sqrt{15}}\arctan\frac{x+\frac{1}{2}}{\sqrt{15/4}} + C \\
&= \frac{1}{2}\ln(x^2+x+4) + \frac{1}{\sqrt{15}}\arctan\frac{2x+1}{\sqrt{15}} + C
\end{aligned}$$

We dropped the absolute value bars from the natural log since its argument was never negative. The final answer is

$$\frac{1}{2}\ln(x^2+x+4)+\frac{1}{\sqrt{15}}\arctan\frac{2x+1}{\sqrt{15}}-\ln|x+2|+C$$

**Example 7.** $\displaystyle\int \frac{3x^4+31x^3+122x^2+219x+159}{x^3+9x^2+27x+27}\,dx$

*Solution:* The first thing we should notice is that the degree of the numerator is **not** less than the degree of the denominator. Applying polynomial long division, we learn that the quotient is $3x+4$ and that remainder is $5x^2+30x+51$. Thus,

$$\frac{3x^4+31x^3+122x^2+219x+159}{x^3+9x^2+27x+27}=3x+4+\frac{5x^2+30x+51}{x^3+9x^2+27x+27}$$

We now find the partial fraction decomposition of the last term. The denominator factors as $(x+3)^3$, and so

$$\frac{5x^2+30x+51}{x^3+9x^2+27x+27}=\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{(x+3)^3}$$

Clearing denominators leads to

$$5x^2+30x+51=A(x+3)^2+B(x+3)+C \tag{4}$$

We can quickly determine $C$ by evaluating at $x=-3$, which leads to $5(-3)^2+30(-3)+51=C$, and so $C=6$. We now pick two simple values of $x$ to obtain relations between $A$ and $B$. From $x=-2$, we find

$$11=A(1)^2+B(1)+6 \qquad\Longrightarrow\qquad 5=A+B$$

and from $x=-4$, we find

$$11=A(-1)^2+B(-1)+6 \qquad\Longrightarrow\qquad 5=A-B$$

Adding these equations together, we find that $10=2A$ and so $A=5$. Substituting this back into $11=A+B$ yields $B=0$. Thus

$$\begin{aligned}
\int \frac{3x^4+31x^3+122x^2+219x+159}{x^3+9x^2+27x+27}\,dx\\
&=\int 3x+4+\frac{5x^2+30x+51}{x^3+9x^2+27x+27}\,dx\\
&=\int 3x+4\,dx+\int \frac{5}{x+3}+\frac{6}{(x+3)^3}\,dx\\
&=\frac{3}{2}x^2+4x+5\ln|x+3|+\frac{6}{-2}\frac{1}{(x+3)^2}+C\\
&=\frac{3}{2}x^2+4x+5\ln|x+3|-\frac{3}{(x+3)^2}+C
\end{aligned}$$

## Exercises

Compute the following integrals using partial fraction expansions.

1. $\displaystyle\int \frac{x}{x^2+4x+4}\,dx$

2. $\displaystyle\int \frac{x-2}{x^2+4x+4}\,dx$

3. $\displaystyle\int \frac{dz}{(z+1)(z+2)(z+3)}$

4. $\displaystyle\int \frac{5dt}{(t+4)^2(t-1)}$

5. $\displaystyle\int \frac{3x^2-x+8}{(x+4)(x^2+4)}\,dx$

6. $\displaystyle\int \frac{5w^2-3w-21}{w^2-w-6}\,dw$

7. $\displaystyle\int \frac{4\sin\theta+21}{(3+\sin\theta)(6+\sin\theta)}\cos\theta\,d\theta$

8. $\displaystyle\int \frac{e^a}{e^{2a}+e^a-20}\,da$