

# Calibrating the Ornstein-Uhlenbeck (Vasicek) model

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In this article I'll describe two methods for calibrating the model parameters of the Ornstein-Uhlenbeck process to a given dataset.

- The least squares regression method
- maximum likelihood method

## Introduction

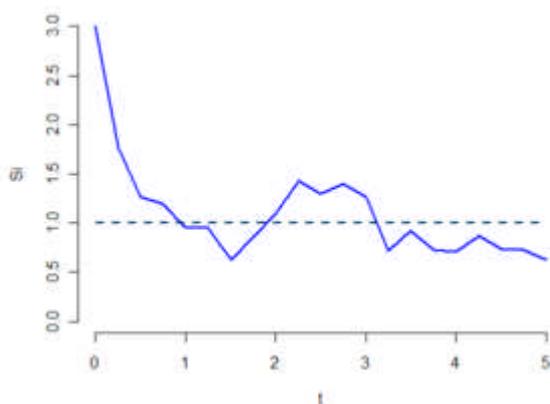
The stochastic differential equation (SDE) for the Ornstein-Uhlenbeck process is given by

$$dS_t = \lambda(\mu - S_t)dt + \sigma dW_t$$

with  $\lambda$  the mean reversion rate,  $\mu$  the mean, and  $\sigma$  the volatility.

## An example simulation

The table and figure below show a simulated scenario for the Ornstein-Uhlenbeck process with time step  $\delta=0.25$ , mean reversion rate  $\lambda=3.0$ , long term mean  $\mu=1.0$  and a noise term of  $\sigma=0.50$ . We will use this data to explain the model calibration steps.



A scenarios of a the Ornstein-Uhlenbeck process. The scenarios start at  $S(0)=3$  and reverting to a long term mean of 1.

<b>i</b>	<b>t</b>	$S_i$	$N_{0,1}$
0	0.00	3.0000	
1	0.25	1.7600	-1.0268
2	0.50	1.2693	-0.4985
3	0.75	1.1960	0.3825
4	1.00	0.9468	-0.8102
5	1.25	0.9532	-0.1206
6	1.50	0.6252	-1.9604
7	1.75	0.8604	0.2079
8	2.00	1.0984	0.9134
9	2.25	1.4310	2.1375
10	2.50	1.3019	0.5461
11	2.75	1.4005	1.4335
12	3.00	1.2686	0.4414
13	3.25	0.7147	-2.2912
14	3.50	0.9237	0.3249
15	3.75	0.7297	-1.3019
16	4.00	0.7105	-0.8995
17	4.25	0.8683	0.0281
18	4.50	0.7406	-1.0959
19	4.75	0.7314	-0.8118
20	5.00	0.6232	-1.3890

The following simulation equation is used for generating paths (sampled with fixed time steps of  $\delta=0.25$ ). The equation is an exact solution of the SDE.

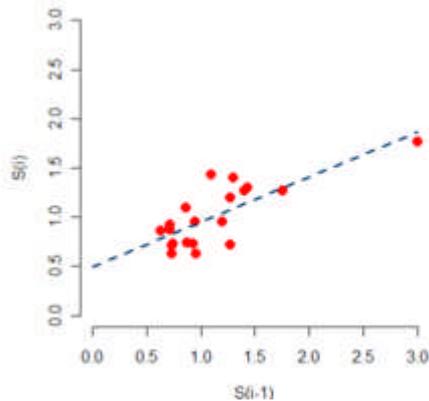
$$S_{i+1} = S_i e^{-\lambda\delta} + \mu(1 - e^{-\lambda\delta}) + \sigma \sqrt{\frac{1 - e^{-2\lambda\delta}}{2\lambda}} N_{0,1}$$

The random numbers used in this example are shown in the last column of the table 1.

## Calibration using least squares regression

The relationship between consecutive observations  $S_i, S_{i+1}$  is linear with a iid normal random term  $\epsilon$

$$S_{i+1} = aS_i + b + \epsilon$$



Least square fitting of a line to the data.

The relationship between the linear fit and the model parameters is given by

$$a = e^{-\lambda\delta}$$
$$b = \mu(1 - e^{-\lambda\delta})$$
$$sd(\epsilon) = \sigma \sqrt{\frac{1 - e^{-2\lambda\delta}}{2\lambda}}$$

rewriting these equations gives

$$\lambda = -\frac{\ln a}{\delta}$$
$$\mu = \frac{b}{1 - a}$$
$$\sigma = sd(\epsilon) \sqrt{\frac{-2 \ln a}{\delta(1 - a^2)}}$$

## Calculating the least squares regression

Most software tools (Excel, Matlab, R, Octave, Maple, ...) have built in functionality for least square regression. If its not available, a least square regression can easily be done by calculating the the quantities below:

$$S_x = \sum_{i=1}^n S_{i-1}$$
$$S_y = \sum_{i=1}^n S_i$$
$$S_{xx} = \sum_{i=1}^n S_{i-1}^2$$
$$S_{xy} = \sum_{i=1}^n S_{i-1}S_i$$
$$S_{yy} = \sum_{i=1}^n S_i^2$$

from which we get the following parameters of the least square fit

$$a = \frac{nS_{xy} - S_x S_y}{nS_{xx} - S_x^2}$$
$$b = \frac{S_y - aS_x}{n}$$
$$sd(\epsilon) = \sqrt{\frac{nS_{yy} - S_y^2 - a(nS_{xy} - S_x S_y)}{n(n-2)}}$$

## Example

Applying the regression to the data in table 1 we get

### Param Value

$S_x$	22.5301
$S_y$	20.1534
$S_{xx}$	30.8338
$S_{xy}$	25.1973
$S_{yy}$	22.2222
$a$	0.4574
$b$	0.4924
$sd(\epsilon)$	0.2073

These results allow us to recover the model parameters:

### Param Value

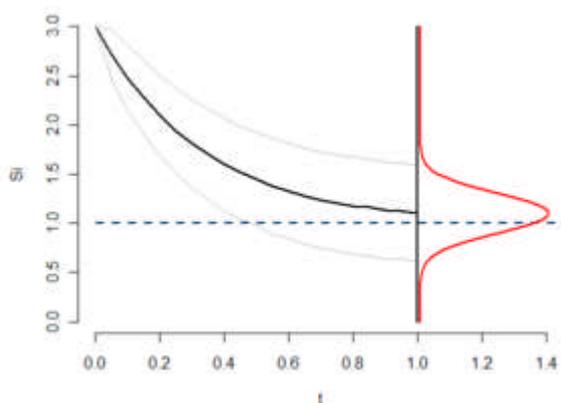
$\mu$	0.9075
$\lambda$	3.1288
$\sigma$	0.5831

## Calibration using Maximum Likelihood estimates

### Conditional probability density function

The conditional probability density function is easily derived by combining the simulation equation above with the probability density function of the normal distribution function:

$$P(N_{0,1} = x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$



Conditional probability density function -red- of S at t=1.

The equation of the conditional probability density of an observation  $S_{i+1}$  given a previous observation  $S_i$  (with a  $\delta$  time step between them) is given by

$$f(S_{i+1}|S_i; \mu, \lambda, \hat{\sigma}) = \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \exp \left[ -\frac{(S_i - S_{i-1}e^{-\lambda\delta} - \mu(1 - e^{-\lambda\delta}))^2}{2\hat{\sigma}^2} \right]$$

with

$$\hat{\sigma}^2 = \sigma^2 \frac{1 - e^{-2\lambda\delta}}{2\lambda}$$

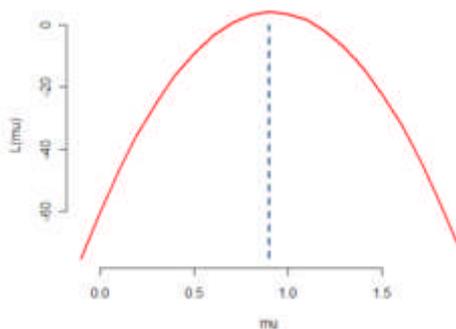
### Log-likelihood function

The log-likelihood function of a set of observation  $(S_0, S_1, \dots, S_n)$  can be derived from the conditional density function

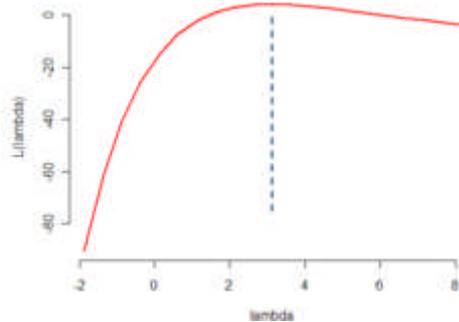
$$\begin{aligned} \mathcal{L}(\mu, \lambda, \hat{\sigma}) &= \sum_{i=1}^n \ln f(S_i|S_{i-1}; \mu, \lambda, \hat{\sigma}) \\ &= -\frac{n}{2} \ln(2\pi) - n \ln(\hat{\sigma}) \\ &\quad - \frac{1}{2\hat{\sigma}^2} \sum_{i=1}^n \left[ S_i - S_{i-1}e^{-\lambda\delta} - \mu(1 - e^{-\lambda\delta}) \right]^2 \end{aligned}$$

### Maximum likelihood conditions

The maximum of this log-likelihood surface can be found at the location where all the partial derivatives are zero. This leads to the following set of constraints.



Log likelihood function as function of mu.



Log likelihood function as function of lambda.

$$\begin{aligned} \frac{\partial \mathcal{L}(\mu, \lambda, \hat{\sigma})}{\partial \mu} &= 0 \\ &= \frac{1}{\hat{\sigma}^2} \sum_{i=1}^n [S_i - S_{i-1}e^{-\lambda\delta} - \mu(1 - e^{-\lambda\delta})] \\ \mu &= \frac{\sum_{i=1}^n [S_i - S_{i-1}e^{-\lambda\delta}]}{n(1 - e^{-\lambda\delta})} \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathcal{L}(\mu, \lambda, \hat{\sigma})}{\partial \lambda} &= 0 \\ &= -\frac{\delta e^{-\lambda\delta}}{\hat{\sigma}^2} \sum_{i=1}^n [(S_i - \mu)(S_{i-1} - \mu) - e^{-\lambda\delta} (S_{i-1} - \mu)^2] \\ \lambda &= -\frac{1}{\delta} \ln \frac{\sum_{i=1}^n (S_i - \mu)(S_{i-1} - \mu)}{\sum_{i=1}^n (S_{i-1} - \mu)^2} \end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}(\mu, \lambda, \hat{\sigma})}{\partial \hat{\sigma}} &= 0 \\
&= \frac{n}{\hat{\sigma}} - \frac{1}{\hat{\sigma}^3} \sum_{i=1}^n \left[ (S_i - \mu - e^{-\lambda \delta} (S_{i-1} - \mu)) \right]^2 \\
\hat{\sigma}^2 &= \frac{1}{n} \sum_{i=1}^n \left[ (S_i - \mu - e^{-\lambda \delta} (S_{i-1} - \mu)) \right]^2
\end{aligned}$$

### Solution of the conditions

The problem with these conditions is that the solutions depend on each other. However, both  $\lambda$  and  $\mu$  are independent of  $\sigma$ , and knowing either  $\lambda$  or  $\mu$  will directly give the value the other. The solution of  $\sigma$  can be found once both  $\lambda$  and  $\mu$  are determined. To solve these equations it is thus sufficient to find either  $\lambda$  or  $\mu$ .

Finding  $\mu$  can be done by substituting the  $\lambda$  condition into the  $\mu$ .

First we change notation of the  $\mu$  and  $\lambda$  condition using the same notation as before, i.e

$$\begin{aligned}
S_x &= \sum_{i=1}^n S_{i-1} \\
S_y &= \sum_{i=1}^n S_i \\
S_{xx} &= \sum_{i=1}^n S_{i-1}^2 \\
S_{xy} &= \sum_{i=1}^n S_{i-1} S_i \\
S_{yy} &= \sum_{i=1}^n S_i^2
\end{aligned}$$

which gives us:

$$\begin{aligned}
\mu &= \frac{S_y - e^{-\lambda \delta} S_x}{n(1 - e^{-\lambda \delta})} \\
\lambda &= -\frac{1}{\delta} \ln \frac{S_{xy} - \mu S_x - \mu S_y + n\mu^2}{S_{xx} - 2\mu S_x + n\mu^2}
\end{aligned}$$

substituting  $\lambda$  into  $\mu$  gives

$$n\mu = \frac{S_y - \left( \frac{S_{xy} - \mu S_x - \mu S_y + n\mu^2}{S_{xx} - 2\mu S_x + n\mu^2} \right) S_x}{1 - \left( \frac{S_{xy} - \mu S_x - \mu S_y + n\mu^2}{S_{xx} - 2\mu S_x + n\mu^2} \right)}$$

removing denominators

$$n\mu = \frac{S_y(S_{xx} - 2\mu S_x + n\mu^2) - (S_{xy} - \mu S_x - \mu S_y + n\mu^2)S_x}{(S_{xx} - 2\mu S_x + n\mu^2) - (S_{xy} - \mu S_x - \mu S_y + n\mu^2)}$$

collecting terms

$$n\mu = \frac{(S_y S_{xx} - S_x S_{xy}) + \mu(S_x^2 - S_x S_y) + \mu^2 n(S_y - S_x)}{(S_{xx} - S_{xy}) + \mu(S_y - S_x)}$$

moving all  $\mu$  to the left

$$n\mu(S_{xx} - S_{xy}) - \mu(S_x^2 - S_x S_y) = S_y S_{xx} - S_x S_{xy}$$

### Final results: The maximum likelihood equations

mean:

$$\mu = \frac{S_y S_{xx} - S_x S_{xy}}{n(S_{xx} - S_{xy}) - (S_x^2 - S_x S_y)}$$

mean reversion rate:

$$\lambda = -\frac{1}{\delta} \ln \frac{S_{xy} - \mu S_x - \mu S_y + n\mu^2}{S_{xx} - 2\mu S_x + n\mu^2}$$

variance:

$$\hat{\sigma}^2 = \frac{1}{n} [S_{yy} - 2\alpha S_{xy} + \alpha^2 S_{xx} - 2\mu(1 - \alpha)(S_y - \alpha S_x) + n\mu^2(1 - \alpha)^2]$$

$$\sigma^2 = \hat{\sigma}^2 \frac{2\lambda}{1 - \alpha^2}$$

with  $\alpha = e^{-\lambda\delta}$

## Example

Calculating the sums based on table 1 we get

### Param Value

$S_x$	22.5301
$S_y$	20.1534
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These results allow us to recover the model parameters:

### Param Value

$\mu$	0.9075
$\lambda$	3.1288
$\sigma$	0.5532